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# $G$-ALGEBRAS, GROUP GRADED ALGEBRAS, AND CLIFFORD EXTENSIONS OF BLOCKS

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Abstract. Let $K$ be a normal subgroup of the finite group $H$. To a block of a $K$-interior $H$-algebra we associate a group extension, and we prove that this extension is isomorphic to an extension associated to a block given by the Brauer homomorphism. This may be regarded as a generalization and an alternative treatment of Dade's results [5, Section 12].

## 1. Introduction

**1.1.** Let $p$ be a prime, and let $\mathcal{O}$ be a discrete valuation ring with residue field $k$ of characteristic $p$. We make no assumptions on the size of $\mathcal{O}$ and $k$, also allowing $\mathcal{O} = k$. Let $K$ be a normal subgroup of the finite group $H$, and we denote by $G$ the factor group $H/K$.

**1.2.** Under the assumption that $k$ is algebraically closed, Dade introduced in [5] the Clifford extension of a block $b$ of $K$, and proved that this extension (which is a central extension of $k^*$ by a certain stabilizer of $b$) is isomorphic to an extension associated to an irreducible modular character lying in a Brauer correspondent of $b$ (see [5, Section 12]). Let us briefly discuss the arguments and assumptions used in [5]. Two main ingredients for the isomorphism of the Clifford extensions were [5, Theorems 8.7 and 9.5]. Note that [5, Theorem 8.7] is essentially Brauer's First Main Theorem, while [5, Theorem 9.5] extends the Brauer correspondence to the case of conjugacy classes of maximal ideals. Besides these two theorems there are two important sets of hypotheses, namely [5, 7.1] and [5, 10.1]. These conditions assure that the equalities [5, 10.4] hold, and give a situation which is a slightly more general case than of the group ring $\mathcal{O}H$.

**1.3.** Here we consider a unitary $K$-interior $H$-algebra $A_1$ over $\mathcal{O}$. The $K$-interior $H$-algebra $A_1$ gives rise to an $H$-interior algebra $A$, which is strongly $G$-graded (see [10, 9.1], or [7, 2.1 and 2.2] for a more general version of this construction). The $H$-interior algebra $A$ does not satisfy conditions [5, 7.1], and therefore the equality [5, 10.4 a)],

$$C_A(A_1) = A^K,$$

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it is also not satisfied. Still, one of the centralizers occuring in this equality is suitable for the construction of the extensions. We will not use a mix of two results analogous to [5, Theorems 8.7 and 9.5], because for that we would need more assumptions. Instead, we consider two blocks that arise from [2, Theorem 3.2].

**1.4.** We start by choosing a primitive idempotent $b$ of $A_1^K$ which lies in the center of $A_1$. By the assumptions made on $A_1$, the idempotent $b$ is actually a block of $A_1$. Instead of working with the centralizer $C_A(A_1)$ we consider $A^K$. As we have already mentioned, the equality $C_A(A_1) = A^K$ holds if the algebra $A$ satisfies assumptions [5, 7.1]. In our case we only have the inclusion

$$C_A(A_1) \subseteq A^K = C_A(K \cdot 1),$$

because $A_1$ is $K$-interior. We work here with $A^K$, and we consider, as in [5, Paragraph 2] the subgroup $G_b$ of $G$ fixing $b$, and its normal subgroup $G[b]$. Restricting $A^K$ to the components indexed by $G[b]$ gives a strongly $G[b]$-graded $G_b$-algebra, and this allows the construction in Section 2 of the Clifford extension associated to the block $b$.

**1.5.** Let $P \leq K$ be a defect group of the point $\{b\}$ of $A_1^K$. The Brauer quotient $A(P)$ of the algebra $A$ with respect to $P$ is a $G$-graded algebra, and its restriction $A'(P)$ to $C_H(P)K/K$ is strongly graded. The Brauer quotient $A_1(P)$ of $A_1$ is a $C_K(P)$-interior $C_H(P)$-algebra, and, as in [7, 2.1] again, it gives rise to a $C_H(P)$-interior $C_H(P)/C_K(P)$-graded algebra $R$. We show in Section 3 that $R$ and $A'(P)$ are isomorphic as $N_H(P)$-algebras.

By [2, Theorem 3.2], to the point $\{b\}$ of $A_1^K$ it corresponds the point $\{\bar{b} := \mathrm{Br}_P(b)\}$ of $A_1(P)^{N_K(P)}$. Since $b$ is central then so is $\bar{b}$. Using the isomorphism stated in Proposition 3.4, we regard $\bar{b}$ as a primitive idempotent of $R_1^{N_K(P)}$. Next, we consider the centralizer

$$C_R(C_K(P))^{N_K(P)} = R^{N_K(P)}.$$

This is a $C_H(P)/C_K(P)$-graded $N_H(P)$-algebra, and we localize it by using $\bar{b}$ to the subgroup $C_H(P)_{\bar{b}}/C_K(P)$. Further, we take the $N_K(P)$-fixed elements and we consider the group $\overline{C_H(P)}[\bar{b}]$ for which $\bar{b}R^{N_K(P)}$ is strongly graded. We construct in Section 4 the first Clifford extension associated to $\bar{b}$.

**1.6.** Of course, the algebra $A(P)$ is $C_H(P)$-interior. The centralizer

$$C_{A(P)}(C_K(P)) = A(P)^{C_K(P)}$$

is too large, so one must take

$$(C_{A(P)}(C_K(P)))^{N_K(P)} = A(P)^{N_K(P)}.$$

Going back to the $G$-graded Brauer quotient we see that

$$\bar{b}A(P)\bar{b} = \bigoplus_{\sigma \in G_b} \bar{b}A(P)_\sigma.$$

We denote $G[\bar{b}]$ the subgroup of $G_b$ such that

$$E := \bigoplus_{\sigma \in G[\bar{b}]} \bar{b}A(P)_\sigma^{N_K(P)}$$

is strongly graded. Since $E_1 = \bar{b}A_1(P)^{N_K(P)}$ is local, we consider the quotient

$$\bar{E} = E/EJ(E_1)$$

the crossed product of $\bar{E}_1$ with $G[\bar{b}]$ corresponding to the second Clifford extension of $\bar{b}$.

**1.7.** In [5, Section 11] a so-called "right kernel" of a bilinear map is introduced. That map exists because of the assumption that the field $k$ is algebraically closed, so the construction of the Clifford extensions yields twisted group algebras over $k$ whose components are one dimensional. In the present situation, we see that in general the two Clifford extensions correspond to crossed products over the skew fields

$$\bar{C}_1 := bA_1^K/J(bA_1^K)$$

and

$$\bar{E}_1 := \bar{b}(A_1(P))^{N_K(P)}/J(\bar{b}(A_1(P))^{N_K(P)})$$

extending $k$, respectively. Clearly $\bar{C}_1$ and $\bar{E}_1$ are trivially $N_K(P)$-acted so there is no need to work with this subgroup.

Our main result is Theorem 7.3 below, where we show that the above crossed products containing $\bar{C}_1$ and $\bar{E}_1$ as identity components are isomorphic as group-graded algebras, and the isomorphism preserves the action of the stabilizer of $b$. Moreover, we show that $G[b] = G[\bar{b}]$ is a subgroup in the normalizer of $P$ in $H_b$ and that these two isomorphic extensions contain the first Clifford extension of $\bar{b}$.

**1.8.** At the end of the paper we deal with the special case of the group algebra. In this case, our $K$-interior $H$-algebra is $A_1 = \mathcal{O}K$, and we have the equalities

$$C_{\mathcal{O}H}(\mathcal{O}K) = (\mathcal{O}H)^K$$

and

$$(C_{kC_H(D)}(kC_K(D)))^{N_K(D)} = (kC_H(D))^{N_K(D)}.$$

For this particular $K$-interior $H$-algebra, the result tells more. For both algebras $\mathcal{O}H^P$ and $\mathcal{O}N_H(P)^P$ the Brauer quotient is the same, exactly $kC_H(P)$. So considering again the Clifford extensions as before one ends up with three isomorphic crossed products acted by the same $N_H(P)_b$. This generalizes [5, Corollay 12.6] (and see also [4] for a concise presentation) to the case of arbitrary base field $k$.

Our general assumptions and notations are standard. We refer the reader to [12] and [10] for Puig's theory of $G$-algebras and pointed groups, and to [9] for facts on group graded algebras.

## 2. The Clifford extension of a block

**2.1.** As in the introduction, let $K$ be a normal subgroup of the finite group $H$, and let $G = H/K$. Let $A_1$ be an unitary $K$-interior $H$-algebra over the $\mathcal{O}$. As is [7, 2.1], there exists a strongly $G$-graded algebra

$$A := \bigoplus_{\sigma \in G} A_\sigma$$

with structural homomorphisms

$$\mathcal{O}H \to A$$

of $G$-graded algebras. This homomorphism endows $A$ with the structure of a $G$-graded $H$-interior algebra, and hence $A$ is also an $H$-algebra by conjugation:

$$a_\sigma^h = h^{-1} \cdot a_\sigma \cdot h \in A_{\sigma^h},$$

where $a_\sigma \in A_\sigma$, $\sigma \in G$, and $h \in H$. Moreover, we have that

$$A_\sigma = A_1 \otimes x \text{ and } A_1 \simeq A_1 \otimes 1,$$

for each $\sigma \in G$. Here $x$ is a representative in $\sigma$ and the tensor product is over $k$. The multiplication in $A$ is defined in [7, 2.1], and see also [10, 9.1].

**2.2.** Let $A_1^K$ denote the subalgebra of $A_1$ consisting of elements fixed under the conjugation action of $K$. The interior $H$-algebra is $K$-interior by restriction, hence in the same manner we may consider

$$C_A(K \cdot 1) = A^K,$$

the fixed elements in $A$ under the conjugation action of the same group $K$. Because $K$ is a normal subgroup of $H$ we clearly have that

$$A^K = \bigoplus_{\sigma \in G} (A_\sigma)^K$$

is a $G$-graded subalgebra of $A$.

**2.3.** Let $b$ be a primitive idempotent of $A_1^K$, lying in $Z(A_1)$. The idempotent $b$ need not necessarily be central in $C_A(K \cdot 1)$, so it is convenient to consider the stabilizer $G_b$ of $b$ in $G$. In this situation, $bA^K$ becomes a $G_b$-graded $G_b$-interior algebra, but it is not strongly graded in general. As in [5], we consider the subset

$$G[b] = \{\sigma \in G_b \mid (bA_\sigma)^K \cdot (bA_{\sigma^{-1}})^K = (bA_1)^K\}.$$

of $G_b$.

**Proposition 2.4.** *The subset $G[b]$ is a normal subgroup of $G_b$.*

*Proof.* For all $\sigma, \tau \in G[b]$, we have

$$\begin{aligned}(bA_1)^K =&(bA_\sigma)^K \cdot (bA_{\sigma^{-1}})^K = (bA_\sigma)^K \cdot (bA_1)^K \cdot (bA_{\sigma^{-1}})^K \\ =&(bA_\sigma)^K \cdot (bA_\tau)^K \cdot (bA_{\tau^{-1}})^K \cdot (bA_{\sigma^{-1}}) \\ \subseteq&(bA_{\sigma\tau})^K \cdot (bA_{(\sigma\tau)^{-1}})^K.\end{aligned}$$

This proves that $G[b]$ is a subgroup of $G_b$. Now consider the elements $\sigma \in G[b]$ and $\theta \in G_b$. We shall prove the equality

$$((bA_\sigma)^K)^\theta = (bA_{\sigma^\theta})^K.$$

For this, it suffices to take $h \in \theta$ and prove that

$$((bA_\sigma)^K)^h = (bA_{\sigma^h})^K.$$

Let $a_\sigma \in (bA_\sigma)^K$, then $a_\sigma^h \in A_{\sigma^h}$. For any $l \in K$ we have

$$(a_\sigma^h)^l = l^{-1} \cdot a_\sigma^h \cdot l = l^{-1}h^{-1} \cdot a_\sigma \cdot hl = h^{-1} \cdot a_\sigma \cdot h.$$

Conversely, let $a_{\sigma^h} \in (bA_{\sigma^h})^K$. As before, we obtain that $(a_{\sigma^h})^{h^{-1}} \in (bA_\sigma)^K$, and by applying the action of $h$, the desired inclusion follows. We have

$$((bA_\sigma)^K)^\theta \cdot ((bA_{\sigma^{-1}})^K)^\theta = ((bA_1)^K)^\theta,$$

or equivalently,

$$(bA_{\sigma^\theta})^K \cdot (bA_{(\sigma^\theta)^{-1}})^K = (bA_1)^K,$$

which proves the statement. □

**2.5.** We now denote
$$C := \bigoplus_{\sigma\in G[b]} b(A_\sigma)^K = \bigoplus_{\sigma\in G[b]} C_\sigma$$
where for each $\sigma \in G[b]$, we have denoted
$$C_\sigma := b(A_\sigma)^K.$$
The proposition above implies that $C$ is a strongly $G[b]$-graded $G_b$-algebra. Its identity component
$$C_1 = bA_1^K$$
is a local ring, so by [11, Lemma 1.1] it follows that $C$ is a crossed product of $C_1$ with $G[b]$.

**2.6.** We have the skew field
$$\bar{C}_1 := C_1/J(C_1)$$
whose center $\hat{k}_1 = Z(\bar{C}_1)$ is a finite extension of $k$. Consequently,
$$\bar{C} := C/CJ(C_1)$$
is a crossed product, and in the same time it is a $G_b$-algebra. For any $\sigma \in G[b]$ we identify
$$\bar{C}_\sigma = C_\sigma/C_\sigma J(C_1).$$
The corresponding group extension
$$1 \to \bar{C}_1^* \to \mathrm{hU}(\bar{C}) \to G[b] \to 1 \tag{1}$$
is the *Clifford extension* of the block $b$. Here hU denotes the group of homogenous units.

## 3. The Brauer Quotient

By the Brauer quotient of a $H$-algebra with respect to a $p$-subgroup $P$ of $H$ we mean the structure of an $N_H(P)$-algebra as presented in [12, §11]. In general, there is no natural graded structure of the Brauer quotient, with respect to an arbitrary $p$-subgroup of $H$, on our strongly $G$-graded $H$-interior algebra $A$. Therefore, we consider the situation when $P$ is a $p$-subgroup of $K$.

**3.1.** Let $P$ be a $p$-subgroup of $K$. Since the restriction to $K$ of the action of $H$ on the algebra $A$ leaves invariant each homogeneous component of $A$, we have
$$A^P = \bigoplus_{\sigma\in G} A_\sigma^P.$$
For any subgroup $Q$ in $P$ we have the equality $A_Q^P \cap A_\sigma^P = (A_\sigma)_Q^P$. Using this equality, the Brauer quotient of $A$ becomes
$$\begin{aligned} A(P) &= k\otimes_{\mathcal{O}} (A^P/\sum_{Q<P} A_Q^P) \\ &= k\otimes_{\mathcal{O}} (\bigoplus_{\sigma\in G}(A_\sigma)^P)/(\bigoplus_{\sigma\in G}\sum_{Q<P}(A_\sigma)_Q^P) \\ &\simeq k\otimes_{\mathcal{O}} \bigoplus_{\sigma\in G}((A_\sigma)^P/\sum_{Q<P}(A_\sigma)_Q^P) = \bigoplus_{\sigma\in G} A(P)_\sigma. \end{aligned}$$
In this situation, the $N_H(P)$-algebra $A(P)$ is $G$-graded, with identity component $A(P)_1 \simeq A_1(P)$.

**3.2.** It is useful to consider the subalgebra

$$A'(P) = \bigoplus_{\sigma \in C_H(P)K/K} A(P)_\sigma,$$

which is an $N_H(P)$-invariant subalgebra of $A(P)$. With the notation of 2.1, the map $a \mapsto a \otimes 1$ is an isomorphism of $H$-algebras between $A_1$ and $A_1 \otimes 1$. This implies that

$$A_1^P \simeq (A_1 \otimes 1)^P = A_1^P \otimes 1.$$

If $\sigma \in C_H(P)K/K$ and $a \in A_1^P$ and $x \in \sigma \cap C_H(P)$, we can identify $a$ with

$$a \otimes 1 = (a \otimes x)(1 \otimes x^{-1}) \in A_\sigma^P \cdot A_{\sigma^{-1}}^P.$$

Then the equality $A_1^P = A_\sigma^P \cdot A_{\sigma^{-1}}^P$ is valid, showing that $A'(P)$ is a strongly $C_H(P)K/K$-graded $N_H(P)$-algebra. In fact, this also follows from the proof of Proposition 3.4 below.

**3.3.** We may also construct the Brauer quotient $A_1(P)$ of the the $K$-interior $H$-algebra $A_1$, so $A_1(P)$ is a $C_K(P)$-interior $N_H(P)$-algebra. We then regard $A_1(P)$ to be a $C_K(P)$-interior $C_H(P)$-algebra. In this case, following [7, 2.1] we obtain the strongly $C_H(P)/C_K(P)$-graded $N_H(P)$-interior $k$-algebra

$$R := \bigoplus_{\tau \in C_H(P)/C_K(P)} R_\tau.$$

For each $\tau \in C_H(P)/C_K(P)$ we have

$$R_\tau = A_1(P) \otimes x$$

for some representative $x$ in $\tau$.

**Proposition 3.4.** *The strongly graded $k$-algebras $A'(P)$ and $R$ are isomorphic as $N_H(P)$-algebras.*

*Proof.* There is an obvious bijection between the sets of components of the two algebras, because of the natural isomorphism

$$C_H(P)K/K \simeq C_H(P)/C_K(P).$$

Next, we fix $\tau \in C_H(P)/C_K(P)$ and $\sigma \in C_H(P)K/K$ such that $\tau = \sigma \cap C_H(P)$, and we define

$$\psi_\tau : R_\tau \to A'(P)_\sigma, \qquad \bar{a} \otimes x \mapsto \overline{a \otimes x},$$

for $x \in \tau$. We have denoted $\bar{a} := \mathrm{Br}_P(a)$ for some element $a$ in $A_1^P$, while $x$ stands for a representative in $\tau$. If $a_1 \in \bar{a}$ then $a_1 - a \in \mathrm{Ker}(\mathrm{Br}_P)$ which means $a_1 - a = \mathrm{Tr}_Q^P(c)$ for some $c \in A_1^Q$. Then

$$(a_1 - a) \otimes x = \mathrm{Tr}_Q^P(c) \otimes x = \mathrm{Tr}_Q^P(c \otimes x) \in \sum_{Q<P} (A_\sigma)_Q^P,$$

since $x \in C_H(P)$. So $\psi_\tau$ is a well-defined map. Moreover, the direct sum of these maps gives the graded homomorphism

$$\psi : R \to A'(P),$$

whose identity component is an isomorphism. Indeed, $A'(P)_1 = A_1(P)$ and $R_1 = A_1(P) \otimes 1 \simeq A_1(P)$, and one can easily prove that $\psi_1$ is both injective and surjective. The statement follows by applying [6, Proposition 2.12]. $\square$

## 4. The second extension

**4.1.** Assume that the subgroup $P$ of $K$ is a defect group of $b$. According to [12, §18], this means $b \in (A_1)_P^K$ and $\mathrm{Br}_P(b) \neq 0$. Since $b$ central and primitive in $A_1^K$, it is actually a point of $K$ on $A_1$ with defect group $P$ (see [12, §3]). Applying [2, Theorem 3.2], the element $\bar{b} = \mathrm{Br}_P(b)$ lies in the center of $A_1(P)^{N_K(P)}$ and forms a singleton, hence a point of $A_1(P)^{N_K(P)}$, also with defect group $P$. The identity component $R_1$ of the algebra $R$ constructed in 3.3 is isomorphic to $A_1(P)$. Using this isomorphism, we regard $\bar{b}$ as an element of $R_1$, so $\bar{b}$ is a point of $R_1^{N_K(P)}$ with defect group $P$.

**4.2.** Let $C_H(P)_{\bar{b}}$ denote the stabilizer of $\bar{b}$ in $C_H(P)$, and let

$$S := \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} R_\tau.$$

Moreover $\bar{b}$ is a central element of $S$ and one can check the equality

$$\bar{b}R\bar{b} := \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} \bar{b}R_\tau$$

Then $\bar{b}R\bar{b}$ is $C_H(P)/C_K(P)$-graded $N_H(P)_{\bar{b}}$-invariant algebra.

**4.3.** Since $N_K(P)$ is a normal subgroup of $N_H(P)_{\bar{b}}$, we may consider the centralizer

$$C_S(C_K(P))^{N_K(P)} \cdot 1) = S^{N_K(P)}.$$

The normal subgroup $N_K(P)$ of $N_H(P)_{\bar{b}}$ centralizes $C_H(P)_{\bar{b}}/C_K(P)$, and $S^{N_K(P)}$ is a $N_H(P)_{\bar{b}}$-algebra in which $\bar{b}$ is central. Then

$$S^{N_K(P)} = \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} (R_\tau)^{N_K(P)}, \text{ and}$$

$$\bar{b}S^{N_K(P)} = (\bar{b}S)^{N_K(P)} = (\bar{b}R\bar{b})^{N_K(P)} = \bigoplus_{\tau \in C_H(P)_{\bar{b}}/C_K(P)} \bar{b}(R_\tau)^{N_K(P)}.$$

Of course we need

$$\overline{C_H(P)}[\bar{b}] = \{\tau \in C_H(P)_{\bar{b}}/C_K(P) \mid \bar{b}(R_\tau)^{N_K(P)} \cdot \bar{b}(R_{\tau^{-1}})^{N_K(P)} = \bar{b}(R_1)^{N_K(P)}\},$$

the normal subgroup of $C_H(P)_{\bar{b}}/C_K(P)$ that makes $\bar{b}S^{N_K(P)}$ strongly graded. Denote

$$\begin{aligned} D &:= \bigoplus_{\tau \in \overline{C_H(P)}[\bar{b}]} \bar{b}(R_\tau)^{N_K(P)} \\ &= \bigoplus_{\tau \in \overline{C_H(P)}[\bar{b}]} D_\tau, \end{aligned}$$

where for each $\tau \in \overline{C_H(P)}[\bar{b}]$ we have

$$D_\tau = \bar{b}(R_\tau)^{N_K(P)}.$$

By construction, it follows that $D_\tau D_{\tau^{-1}} = D_1$ for all $\tau \in \overline{C_H(P)}[\bar{b}]$. This makes $D$ a $\overline{C_H(P)}[\bar{b}]$-strongly graded $N_H(P)_{\bar{b}}$-algebra. By the localness of $D_1$ and by [11, Lemma 1.1] the order $D$ is a crossed product of $D_1$ with $\overline{C_H(P)}[\bar{b}]$.

**4.4.** Denote
$$\bar{D}_1 := D_1/J(D_1)$$
and
$$\bar{D} = D/DJ(D_1).$$
Since the identity component $D_1 = \bar{b}R_1^{N_K(P)}$ of $D$ is a local ring, we obtain $\bar{D}_1$ a skew field whose center $\hat{k}_2 = Z(\bar{D}_1)$ is a finite extension of $k$. Moreover, $\bar{D}$ is an $N_H(P)_{\bar{b}}$-algebra, and in the same time a crossed product of $\bar{D}_1$ with $\overline{C_H(P)}[\bar{b}]$, corresponding to the group extension
$$1 \to \bar{D}_1^* \to \mathrm{hU}(\bar{D}) \to \overline{C_H(P)}[\bar{b}] \to 1. \tag{2}$$
As usual, for $\tau \in \overline{C_H(P)}[\bar{b}]$ we identify
$$\bar{D}_\tau = D_\tau/D_\tau J(D_1).$$

## 5. The third extension

If, as above, $N_H(D)_b$ denotes the stabilizer of $b$ in $N_H(P)$ and $N_H(D)_{\bar{b}}$ denotes the stabilizer of $\bar{b}$ in $N_H(P)$, one can easily show that these two stabilizers are equal.

**5.1.** We return to our $G$-graded $H$-interior algebra
$$A = \bigoplus_{\sigma \in G} A_\sigma,$$
and to its $H_b$-subalgebra
$$bA := \bigoplus_{\sigma \in G_b} bA_\sigma.$$
Since the defect group $P$ fixes $b$ we have $(bA)^P = bA^P$ and we may consider the Brauer homomorphism in the following situation.
$$\mathrm{Br}_P : (bA)^P \to (bA)(P).$$
Of course this is a morphism of $N_H(P)_b$-algebras. Note that we have the direct sums decompositions
$$(bA)^P = \bigoplus_{\sigma \in G_b} (bA_\sigma)^P$$
and
$$(bA)(P) = \bigoplus_{\sigma \in G_b} ((bA_\sigma)^P / \sum_{Q<P} (bA_\sigma)_Q^P) := \bigoplus_{\sigma \in G_b} (bA)(P)_\sigma.$$
This is actually the same situation as in Section 3, so we do not give any further explanation on the structure of these algebras.

**5.2.** We are going to relate this Brauer quotient to the construction made in Section 4. For that we need to make one replacement. Recall that $\bar{b}$ denotes the image of $b$ under the Brauer morphism evaluated on the identity components. We see that for any $\sigma \in G_b$ there is an isomorphism of $N_H(P)_b$-invariant $k$-spaces, that is
$$(bA_\sigma)^P / \sum_{Q<P} (bA_\sigma)_Q^P \simeq \bar{b}((A_\sigma)^P / \sum_{Q<P} (A_\sigma)_Q^P).$$
Thus we can reconsider the Brauer morphism in this way:
$$\mathrm{Br}_P : bA^P \to \bar{b}A''(P) = \bar{b}A(P)\bar{b},$$

where
$$A''(P) = \bigoplus_{\sigma\in G_b} A(P)_\sigma.$$

**5.3.** Next we are interested in the subgroups of $G_b$ for which $bA^P$ and $\bar{b}A''(P)$ become strongly graded. So we consider
$$T_1 = \{\sigma \in G_b \mid bA^P_\sigma \cdot bA^P_{\sigma^{-1}} = bA^P_1\}$$
and
$$T_2 = \{\sigma \in G_b \mid \bar{b}A''(P)_\sigma \cdot \bar{b}A''(P)_{\sigma^{-1}} = \bar{b}A(P)_1\}.$$
Since we are concerned mainly with subalgebras that are strongly graded, as we will see, all the subgroups of $G_b$ that appear in our further constructions are included in $T_1$ or in $T_2$. An obvious remark is that the Brauer morphism carries the surjection componentwise, and this implies that the restriction
$$\bigoplus_{\sigma\in T_1}(bA_\sigma)^P \to \bigoplus_{\sigma\in T_2}\bar{b}A(P)_\sigma$$
is well defined. Indeed, if $\sigma \in T_1$ then we have
$$\bar{b}A(P)_1 = \mathrm{Br}_P(bA^P_1) = \mathrm{Br}_P(bA^P_\sigma \cdot bA^P_{\sigma^{-1}}) = \bar{b}A''(P)_\sigma \cdot \bar{b}A''(P)_{\sigma^{-1}}.$$
This restriction could be surjective provided that the idempotent $b$ remains primitive in $A^P_1$, which in general it is not true.

Following the proof of Proposition 2.4 one can show that $T_1$ and $T_2$ are both respectively $N_H(P)_b$-invariant and $N_H(P)_{\bar{b}}$-invariant subgroups of $G_b$. But since $N_H(P)_b = N_H(P)_{\bar{b}}$ and $T_1 \le T_2$ this restriction is a morphism of $N_H(P)_b$-algebras.

**5.4.** At the beginning of Section 4 we saw that $\bar{b}$ is a primitive central idempotent of $A(P)_1^{N_K(P)} \simeq A_1(P)^{N_K(P)}$, hence $\bar{b}$ is a central idempotent of
$$\bar{b}A''(P)^{N_K(P)} = \bigoplus_{\sigma\in G_b}\bar{b}A(P)_\sigma^{N_K(P)}$$
and of
$$(\bigoplus_{\sigma\in T_2}\bar{b}A(P)_\sigma)^{N_K(P)} = \bigoplus_{\sigma\in T_2}\bar{b}A(P)_\sigma^{N_K(P)}.$$
In both cases we can introduce the subgroups
$$G[\bar{b}] = \{\sigma \in G_b \mid \bar{b}A(P)_\sigma^{N_K(P)} \cdot \bar{b}A(P)_{\sigma^{-1}}^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)}\}$$
and
$$T_2[\bar{b}] = \{\sigma \in T_2 \mid \bar{b}A(P)_\sigma^{N_K(P)} \cdot \bar{b}A(P)_{\sigma^{-1}}^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)}\}$$
of $G_b$, determining two strongly graded subalgebras. As expected, these subgroups coincide and are $N_H(P)_b$-invariant.

The last statement follows by using the localness of $\bar{b}A(P)_1^{N_K(P)}$, the inclusions
$$(\bigoplus_{\sigma\in T_2}\bar{b}A(P)_\sigma)^{N_K(P)} \hookrightarrow \bar{b}A''(P)^{N_K(P)} \hookrightarrow \bar{b}A''(P)$$
and [11, Lemma 1.1]. Then for any $\sigma \in T_2[\bar{b}]$ there is a unit $\bar{a}_\sigma \in \bar{b}A(P)_\sigma^{N_K(P)} \cap \mathrm{U}(\bar{b}A''(P)^{N_K(P)})$ such that
$$\bar{b}A(P)_\sigma^{N_K(P)} = \bar{a}_\sigma\bar{b}A(P)_1^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)}\bar{a}_\sigma.$$

So $T_2[\bar{b}]$ is a subgroup of $G[\bar{b}]$. Now let $\sigma \in G[\bar{b}]$. A similar argument as before gives a unit $\bar{a}_\sigma \in \bar{b}A(P)_\sigma^{N_K(P)} \cap \mathrm{U}(\bar{b}A''(P)^{N_K(P)})$ such that

$$\bar{b}A(P)_\sigma^{N_K(P)} = \bar{a}_\sigma \bar{b}A(P)_1^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)}\bar{a}_\sigma,$$

as a component of $\bar{b}A''(P)^{N_K(P)}$. Using one of the above inclusion it follows that $\bar{a}_\sigma$ is an invertible element in the biggest algebra. This implies that $\sigma$ belongs to $T_2$. Keeping in mind that $\bar{a}_\sigma$ is still an $N_K(P)$-invariant invertible element, we have $\sigma \in T_2[\bar{b}]$. The $N_H(P)_b$-invariance follows again using the technique of the proof of Proposition 2.4.

**5.5.** In the previous paragraph we saw that regardless the starting point, we end up with the same subgroup that makes our $N_K(P)$-invariant subalgebra of $\bar{b}A(P)''$ strongly graded. Repeating the construction of Section 4 we obtain the algebra

$$E := \bigoplus_{\sigma \in G[\bar{b}]} \bar{b}A(P)_\sigma^{N_K(P)} := \bigoplus_{\sigma \in G[\bar{b}]} E_\sigma,$$

which is a crossed product of $E_1 := \bar{b}A(P)_1^{N_K(P)}$ with $G[\bar{b}]$ such that $\bar{E}_1 := E_1/J(E_1)$ is a skew field whose center $\hat{k}_3 = Z(\bar{E}_1)$ is a finite extension of $k$. The quotient $\bar{E} := E/EJ(E_1)$ is a $N_H(P)_b$-algebra, a crossed product of $\bar{E}_1$ with $G[\bar{b}]$ corresponding to the Clifford extension

$$1 \to \bar{E}_1^* \to \mathrm{hU}(\bar{E}) \to G[\bar{b}] \to 1. \tag{3}$$

We should note that $E$ does not contain any zero components, meaning that for any $\sigma \in G[\bar{b}]$ we have $\bar{b}A(P)_\sigma^{N_K(P)} \neq 0$.

## 6. Remarks on the first extension

We go back to our first extension, since we have introduced the group $T_1$, and we need to relate it to $G[b]$.

**6.1.** First, let us observe that $G[b]$ is a subgroup of $T_1$. This follows from the fact that $C$ is a crossed product (see 2.5) and because of the inclusions

$$C = \bigoplus_{\sigma \in G[b]} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in G_b} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in G_b} b(A_\sigma)^P.$$

**6.2.** Conversely, as in the previous section, since

$$\bigoplus_{\sigma \in T_1} b(A_\sigma)^K \hookrightarrow \bigoplus_{\sigma \in T_1} b(A_\sigma)^P,$$

we have $T_1[b] = G[b]$.

**6.3.** If $\sigma \in G[b]$ since $bA_1^K = bA_\sigma^K \cdot bA_{\sigma^{-1}}^K$ then $bA_\sigma^K \neq 0$ and $bA_\sigma^K \nsubseteq \sum_{Q<P}(bA_\sigma)_Q^P$. This easily follows from the definition of $G[b]$ and from the proof of Lemma 7.1 below. This remark forces the corresponding component in the Brauer quotient to satisfy $\bar{b}A(P)_\sigma^{N_K(P)} \neq 0$.

## 7. The isomorphism of two extensions

We keep the notations of the previous sections. The next lemma will be needed.

**Lemma 7.1.** *The skew fields $\bar{C}_1$ and $\bar{E}_1$ are isomorphic as $N_H(P)_b$-algebras.*

*Proof.* Recall that
$$\bar{C}_1 = C_1/J(C_1) = bA_1^K/J(bA_1^K)$$
and
$$\bar{E}_1 = E_1/J(E_1) = \bar{b}(A_1(P))^{N_K(P)}/J(\bar{b}(A_1(P))^{N_K(P)}).$$
We have $P$ a defect group of the point $\{b\} \subset A_1^K$, and $b \in (A_1)_P^K$. Arguments similar to those of [8, Lemma 3.4], together with the proof of [3, Lemma 1.12], show that the map
$$\mathrm{Br}_P : bA_1^K \to (\mathrm{Br}_P(b)A_1(P))^{N_K(P)} = \bar{b}(A_1(P))^{N_K(P)}$$
is onto. Notice that
$$\mathrm{Br}_P((A_1)_P^K) = A_1(P)_P^{N_K(P)}.$$
This is true since for $\mathrm{Tr}_P^K(a) \in (A_1)_P^K$, by using the Mackey decomposition, we get
$$\mathrm{Br}_P(\mathrm{Tr}_P^K(a)) = \mathrm{Br}_P(\sum_{l\in P\backslash K/P} \mathrm{Tr}_{P\cap P^l}^P(a^l)) = \sum_{l\in N_K(P)/P} (\mathrm{Br}_P(a))^l.$$
The idempotent $b$ belongs to $(A_1)_P^K$, and the ideal $b(A_1)_P^K$ is mapped onto the ideal $\bar{b}A_1(P)_P^{N_K(P)}$, which contains the identity $\bar{b}$ of $\bar{b}A_1(P)^{N_K(P)}$. To finish the proof, it suffices to apply [10, Proposition 3.23]. □

Before stating the main result we introduce one more subgroup of $T_2$.

**7.2.** The $N_H(P)_b$-subalgebra
$$\bigoplus_{\sigma\in N_H(P)_bK/K} \bar{b}A(P)_\sigma^{N_K(P)}$$
need not be a strongly graded subalgebra of $\bar{b}A''(P)$. So we introduce the $N_H(P)_{\bar{b}}$-invariant subgroup
$$\overline{N_H(P)}[\bar{b}] = \{\sigma \in N_H(P)_bK/K \mid \bar{b}A(P)_\sigma^{N_K(P)} \cdot \bar{b}A(P)_{\sigma^{-1}}^{N_K(P)} = \bar{b}A(P)_1^{N_K(P)}\}$$
of $N_H(P)_bK/K$. The inclusions
$$\bigoplus_{\sigma\in\overline{N_H(P)}[\bar{b}]} \bar{b}A(P)_\sigma^{N_K(P)} \hookrightarrow \bigoplus_{\sigma\in N_H(P)_bK/K} \bar{b}A(P)_\sigma^{N_K(P)} \hookrightarrow \bar{b}A''(P),$$
show that $\overline{N_H(P)}[\bar{b}]$ is a subgroup of $T_2$.

**Theorem 7.3.** *The following statements hold.*

(1) *$G_b$ equals $N_H(P)_bK/K = N_H(P)_{\bar{b}}K/K$.*
(2) *The groups $G[b]$ and $G[\bar{b}]$ are equal and they both coincide with $\overline{N_H(P)_b}[\bar{b}]$.*
(3) *The extensions* (1) *and* (3) *are isomorphic.*
(4) *The isomorphism between the extensions* (1) *and* (3) *is compatible with the identity isomorphism*
$$G[b] \to \overline{N_H(P)}[\bar{b}],$$
*and preserves the action of*
$$G_b \simeq N_H(P)_{\bar{b}}/N_K(P)$$

*on the two extensions.*

(5) *There is a monomorphism from extension* (2) *into extension* (3)*, which is also compatible with the natural monomorphism*

$$\overline{C_H(P)}[\bar{b}] \to G[b],$$

*and preserves the action of* $G_b$ *on these extensions.*

*Proof.* We know that the Brauer morphism is compatible with the $N_H(P)$-action. We also know that $\bar{b} = \mathrm{Br}_P(b)$ and $N_H(P)_b = N_H(P)_{\bar{b}}$. Denote by $H_b$ the inverse image of $G_b$ in $H$. Then $H_b$ normalizes $K$, and since all defect groups of $b$ are conjugate under $K$, $H_b$ acts on the set of defect groups of $b$. We have $b \in (A_1)_P^K$ and $\mathrm{Br}_P(b) \neq 0$. Then if $h \in H_b$, we obtain $b \in (A_1)_{P^h}^K$, and clearly $\mathrm{Br}_{P^h}(b) \neq 0$, because otherwise $b \in \sum_{Q<P^h}(A_1)_Q^{P^h}$, implying $\mathrm{Br}_P(b) = 0$. So for any $h \in H_b$ there is $l \in K$ such that $P^h = P^l$, and from this we obtain $hl^{-1} \in N_{H_b}(P) = N_H(P)_b$. The inclusion $N_H(P)_b K \subseteq H_b$ is trivial, proving the equality

$$H_b = N_H(P)_b K = N_H(P)_{\bar{b}} K.$$

This is our first statement.

We claim that the restriction of the Brauer homomorphism

$$(bA)^K = \bigoplus_{\sigma\in G_b}(bA_\sigma)^K \to \bar{b}A''(P)^{N_K(P)} = \bigoplus_{\sigma\in G_b}\bar{b}A(P)_\sigma^{N_K(P)}$$

is an epimorphism. Indeed, because $P$ is a defect group of $b$, this block belongs to the ideal $(A_1)_P^K \subseteq A_P^K$. The equality

$$\mathrm{Br}_P(bA_P^K) = \bar{b}A''(P)_P^{N_K(P)}$$

is well known. The sets $bA_P^K$ and $\bar{b}A''(P)_P^{N_K(P)}$ are ideals, one in $bA^K$ and the other in $\bar{b}A''(P)^{N_K(P)}$, both containing the identity element of the respective algebra, so the claim is proved.

If $\sigma \in G[b]$ then $bA_1^K = bA_\sigma^K \cdot bA_{\sigma^{-1}}^K$. Using the proof of Lemma 7.1 we obtain

$$\begin{aligned}\bar{b}A(P)_1^{N_K(P)} &= \mathrm{Br}_P(bA_1^K) = \mathrm{Br}_P(bA_\sigma^K)\,\mathrm{Br}_P(bA_{\sigma^{-1}}^K)\\ &= \bar{b}A(P)_\sigma^{N_K(P)}\bar{b}A(P)_{\sigma^{-1}}^{N_K(P)} = E_\sigma \cdot E_{\sigma^{-1}},\end{aligned}$$

so the image of $C$ is in $E$, and $G[b] \leq G[\bar{b}]$.

If $\sigma \in G[\bar{b}]$ we have

$$E_\sigma \cdot E_{\sigma^{-1}} = E_1.$$

Then

$$\mathrm{Br}_P^{-1}(E_\sigma)\cdot \mathrm{Br}_P^{-1}(E_{\sigma^{-1}}) \nsubseteq J(C_1),$$

because otherwise

$$E_\sigma \cdot E_{\sigma^{-1}} = \mathrm{Br}_P(\mathrm{Br}_P^{-1}(E_\sigma)\cdot \mathrm{Br}_P^{-1}(E_{\sigma^{-1}})) \subseteq \mathrm{Br}_P(J(C_1)) \subseteq J(E_1)$$

which is false. We have

$$\mathrm{Br}_P^{-1}(E_\sigma)\cdot \mathrm{Br}_P^{-1}(E_{\sigma^{-1}}) = bA_\sigma^K \cdot bA_{\sigma^{-1}}^K \subseteq C_1,$$

and we obtain

$$J(C_1) + bA_\sigma^K \cdot bA_{\sigma^{-1}}^K = C_1.$$

Consequently, $\sigma \in G[b]$, and this proves the inclusion $G[\bar{b}] \leq G[b]$. Together with this inclusion we have shown that the restriction

$$\mathrm{Br}_P : C \to E$$

is also an epimorphism of $N_H(P)_b$-algebras.

We have already used the inclusion $\mathrm{Br}_P(J(C_1)) \subseteq J(E_1)$, which is a well-known result on $\mathcal{O}$-algebras related by an epimorphism. Using the fact that $C$ and $E$ are strongly graded algebras, the results in [9, 1.5.A.] prove

$$\mathrm{Br}_P(CJ(C_1)) \subseteq EJ(E_1).$$

In this way we get a new $N_H(P)_b$-algebra epimorphisms, namely

$$\overline{\mathrm{Br}_P} : \bar{C} \to \bar{E}, \text{ for } \bar{a} \in \bar{C} \text{ we have } \overline{\mathrm{Br}_P}(\bar{a}) = \overline{\mathrm{Br}_P(a)}.$$

Statement (3) follows from this $G[b]$-graded epimorphism, from the equality $G[b] = G[\bar{b}]$ and Lemma 7.1.

Using (1) the inclusion

$$\bigoplus_{\sigma \in N_H(P)_b K/K} \bar{b}A(P)_\sigma \hookrightarrow \bar{b}A''(P) = \bigoplus_{\sigma \in G_b} \bar{b}A(P)_\sigma$$

is now an equality. Forcing

$$\bigoplus_{\sigma \in N_H(P)_b K/K} \bar{b}A(P)_\sigma^{N_K(P)} \hookrightarrow \bar{b}A''(P)^{N_K(P)} = \bigoplus_{\sigma \in G_b} \bar{b}A(P)_\sigma^{N_K(P)}$$

to be an equality. The definitions of $G[\bar{b}]$ and $\overline{N_H(P)}[\bar{b}]$, see 5.4 and 7.2, prove the equality of these two groups. This shows the last part of (2).

By the construction of the first extension $G[b]$ is a normal subgroup of $G_b$, so by using assertions (1) and (2), $G[\bar{b}]$ is a normal subgroup of $N_H(P)_b K/K$. Since the Brauer map is a morphism of $N_H(P)_b$-algebras, statement (4) of the theorem is immediate.

By Proposition 3.4, the $N_H(P)$-algebras

$$R := \bigoplus_{\tau \in C_H(P)/C_K(P)} R_\tau$$

and

$$A'(P) = \bigoplus_{\sigma \in C_H(P)K/K} A(P)_\sigma$$

are isomorphic as $N_H(P)$-algebras. So $R^{N_K(P)}$ and $A'(P)^{N_K(P)}$ are isomorphic as $N_H(P)$-algebras too. This implies that

$$(\bar{b}R\bar{b})^{N_K(P)} = \bar{b}S^{N_K(P)} \overset{\psi}{\simeq} \bar{b}A'(P)^{N_K(P)}$$

and

$$\bar{b}A'(P)^{N_K(P)} = \bigoplus_{\sigma \in C_H(P)_{\bar{b}} K/K} \bar{b}A'(P)_\sigma^{N_K(P)}.$$

Using the isomorphism $\psi$, the group $\overline{C_H(P)}[\bar{b}]$ is isomorphic to its analogous subgroup of $C_H(P)_{\bar{b}}K/K$. Indeed, if $\overline{C_H(P)}[\bar{b}]'$ denotes the subgroup of $C_H(P)_{\bar{b}}K/K$ isomorphic to $\overline{C_H(P)}[\bar{b}]$ under

$$C_H(P)_{\bar{b}}/C_K(P) \simeq C_H(P)_{\bar{b}}K/K$$

then $\overline{C_H(P)}[\bar b]'$ is the largest subgroup for which the restriction of $\bar bA'(P)^{N_K(P)}$ is strongly graded. If $\tau \in \overline{C_H(P)}[\bar b]$ then

$$\begin{aligned}\bar bA(P)_1^{N_K(P)} = \psi(\bar bR_1^{N_K(P)}) &= \psi(\bar bR_\tau^{N_K(P)}) \cdot \psi(\bar bR_{\tau^{-1}}^{N_K(P)})\\ &= \bar bA(P)_\sigma^{N_K(P)} \cdot \bar bA(P)_{\sigma^{-1}}^{N_K(P)},\end{aligned}$$

where $\sigma \cap C_H(P)_{\bar b} = \tau$ and $\sigma$ belongs to $\overline{C_H(P)}[\bar b]'$. Conversely, if $\sigma \in \overline{C_H(P)}[\bar b]'$, as before

$$\psi^{-1}(\bar bA(P)_\sigma^{N_K(P)}) \cdot \psi^{-1}(\bar bA(P)_{\sigma^{-1}}^{N_K(P)}) \nsubseteq J(\bar bR_1^{N_K(P)}),$$

since $\bar bR_1^{N_K(P)}$ is local and $\psi(J(\bar bR_1^{N_K(P)})) = J(\bar bA_1(P)^{N_K(P)})$. Hence for the corresponding $\tau = \sigma \cap C_H(P)_{\bar b}$ we have

$$\bar bR_1^{N_K(P)} = \bar bR_\tau^{N_K(P)} \cdot \bar bR_{\tau^{-1}}^{N_K(P)}.$$

By the definition of the action of $N_H(P)_b$ on these algebras, the groups $\overline{C_H(P)}[\bar b]'$ and $\overline{C_H(P)}[\bar b]$ are both $N_H(P)_b$-invariant hence normal subgroups of $N_H(P)_bK/K$ and of $N_H(P)_b/C_K(P)$ respectively. So $\overline{C_H(P)}[\bar b]$ embeds into $G[b]$ and then $D$ embeds in $E$. The equalities

$$J(D) = DJ(D_1), J(E) = EJ(E_1) \text{ and } J(E_1) = J(D_1)$$

define a map $\bar D \to \bar E$ such that all the vertical maps in the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \bar D_1^* \arrow[r] \arrow[d] & \mathrm{hU}(\bar D) \arrow[r] \arrow[d] & \overline{C_H(P)}[\bar b] \arrow[r] \arrow[d] & 1\\
1 \arrow[r] & \bar E_1^* \arrow[r] & \mathrm{hU}(\bar E) \arrow[r] & G[\bar b] \arrow[r] & 1
\end{tikzcd}$$

are injective. □

## 8. The group algebra case

**8.1.** The main theorem shows that the group $G[b]$ defining the Clifford extension equals $\overline{N_H(P)_b}[\bar b]$ and it is at least $\overline{C_H(P)}[\bar b]$. This situation is generated by the groups $T_1$ and $T_2$. They are the first that strongly graduate the Brauer domain and codomain, and they include all the others subgroups of the algebras we work with. Actually $G[b]$ equals $\overline{C_H(P)}[\bar b]$ exactly when $\overline{C_H(P)}[\bar b] = \overline{N_H(P)}[\bar b]$.

There are some cases of $K$-interior $H$-algebras for which $T_2$ is included or it equals the centralizer. This is the case of a group algebra. Let $G$ denote the quotient $H/K$, let $A = \mathcal{O}H$ and let $b$ be a block of $\mathcal{O}K$ having defect group $P \leq K$. The special case of the group algebra gives

$$A = \bigoplus_{\sigma\in G} A_\sigma, \text{ where } A_\sigma = \mathcal{O}\sigma, \text{ for all } \sigma \in G$$

and $b$ is primitive in $Z(\mathcal{O}K) = \mathcal{O}K^K$. For each $\sigma \in G$ we denote by $C_\sigma$ the intersection $b(\mathcal{O}\sigma \cap A^K)$. As above we introduce $G[b]$ and $G_b$, and then

$$bAb = \bigoplus_{\sigma\in G_b} bA_\sigma = b\mathcal{O}H_b,$$

while

$$C = \bigoplus_{\sigma\in G[b]} bC_\sigma$$

is a strongly $G[b]$-graded $H_b$-algebra. Letting $\hat{k}_1 = C_1/J(C_1)$, the quotient

$$C/CJ(C_1) = \bigoplus_{\sigma\in G[b]} C_\sigma/C_\sigma J(C_1)$$

is the crossed product of $\hat{k}_1$ with $G[b]$ that corresponds to the Clifford extension

$$1 \to \hat{k}_1^* \to \mathrm{hU}(\bar{C}) \to G[b] \to 1. \tag{1'}$$

**8.2.** If $b_1$ is the Brauer correspondent of $b$, it also has defect $P$ and it lies in

$$Z(\mathcal{O}N_K(P)) = \mathcal{O}N_K(P)^{N_K(P)}.$$

The $N_K(P)$-interior $N_H(P)$-algebra $\mathcal{O}N_K(P)$ is the identity component of

$$\mathcal{O}N_H(P) = \bigoplus_{\sigma\in N_H(P)/N_K(P)} \mathcal{O}\sigma.$$

Moreover

$$b_1\mathcal{O}N_H(P)b_1 = b_1\mathcal{O}N_H(P)_{b_1} = \bigoplus_{\sigma\in N_H(P)_{b_1}/N_K(P)} b_1\mathcal{O}\sigma.$$

As in the above construction, we have the normal subgroup $G[b_1]$ of $N_H(P)_{b_1}/N_K(P)$ determining a strongly graded $N_H(P)_{b_1}$-subalgebra of

$$b_1\mathcal{O}N_H(P)_{b_1}^{N_K(P)} = \bigoplus_{\sigma\in N_H(P)_{b_1}/N_K(P)} b_1(\mathcal{O}\sigma)^{N_K(P)};$$

more precisely,

$$E = \bigoplus_{\sigma\in G[b_1]} E_\sigma = \bigoplus_{\sigma\in G[b_1]} b_1(\mathcal{O}\sigma)^{N_K(P)}.$$

Then letting $\hat{k}_2 = E_1/J(E_1)$, the quotient $\bar{E} = E/EJ(E_1)$ is the crossed product of $\hat{k}_2$ with $G[b_1]$ associated to the extension

$$1 \to \hat{k}_2^* \to \mathrm{hU}(\bar{E}) \to G[b_1] \to 1. \tag{2'}$$

**8.3.** Let us take a look of the Brauer quotients of these two algebras. We have

$$\mathrm{Br}_P^H : (\mathcal{O}H)^P \to (\mathcal{O}H)^P / \sum_{Q<P} (\mathcal{O}H)_Q^P \simeq kC_H(P)$$

and

$$\mathrm{Br}_P^{N_H(P)} : (\mathcal{O}N_H(P))^P \to (\mathcal{O}N_H(P))^P / \sum_{Q<P} (\mathcal{O}N_H(P))_Q^P \simeq kC_H(P).$$

Then we have the maps

$$\mathrm{Br}_P^H : (b\mathcal{O}H_b)^K \to \bar{b}kC_H(P)_{\bar{b}}^{N_K(P)}$$

and

$$\mathrm{Br}_P^{N_H(P)} : (b_1\mathcal{O}N_H(P)_{b_1})^{N_K(P)} \to \bar{b}_1kC_H(P)_{\bar{b}_1}^{N_K(P)},$$

where $\bar{b} := \mathrm{Br}_P^H(b)$, $\bar{b}_1 := \mathrm{Br}_P^{N_H(P)}(b_1)$, $N_H(P)_b = N_H(P)_{\bar{b}}$ and $N_H(P)_{b_1} = N_H(P)_{\bar{b}_1}$. Since $\bar{b} = \bar{b}_1$ (see the proof of [1, Theorem 5.1]), the $N_H(P)_b$-algebra

$$D := \bar{b}kC_H(P)_{\bar{b}}^{N_K(P)}$$

determines a unique extension

$$1 \to \hat{k}_3^* \to \mathrm{hU}(\bar{D}) \to \overline{C_H(P)}[\bar{b}] \to 1. \tag{3'}$$

By applying twice the main result, once for each of the Brauer maps defined above, we get that the extensions $(1')$, $(2')$ and $(3')$ are isomorphic. Note that in the case of the group algebra we have

$$(b\mathcal{O}H_b)(P) = (b\mathcal{O}N_H(P)_bK)(P) = (b_1\mathcal{O}N_H(P)_{b_1})(P) = \bar{b}kC_H(P)_{\bar{b}},$$

and this is why $G[b] = G[\bar{b}] = G[b_1] = \overline{N_H(P)_b}[\bar{b}] = \overline{C_H(P)_b}[\bar{b}]$.

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